

SCHRÖDINGER FLOW INTO ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. We present a short-time existence theorem of solutions to the initial value problem for Schrödinger maps of a closed Riemannian manifold to a compact almost Hermitian manifold. The classical energy method cannot work for this problem since the almost complex structure of the target manifold is not supposed to be parallel with respect to the Levi-Civita connection. In other words, a loss of one derivative arises from the covariant derivative of the almost complex structure. To overcome this difficulty, we introduce a bounded pseudodifferential operator acting on sections of the pullback bundle, and essentially eliminate the loss of one derivative from the partial differential equation of the Schrödinger map.

1. INTRODUCTION

Let (M, g) be an m -dimensional closed Riemannian manifold with a Riemannian metric g , and let (N, J, h) be a $2n$ -dimensional compact almost Hermitian manifold with an almost complex structure J and a Hermitian metric h . Consider the initial value problem for Schrödinger maps $u : \mathbb{R} \times M \rightarrow N$ of the form

$$\frac{\partial u}{\partial t} = J_u \tau(u) \quad \text{in } \mathbb{R} \times M, \quad (1)$$

$$u(0, x) = u_0(x) \quad \text{in } M, \quad (2)$$

where $t \in \mathbb{R}$ is the time variable, $x \in M$, $\partial u / \partial t = du(\partial / \partial t)$, du is the differential of the mapping u , u_0 is a given map of M to N , $\tau(u) = \text{trace } \nabla du$ is the tension field of the map $u(t) : M \rightarrow N$, and ∇ is the induced connection. Here we observe local expression of $\tau(u)$. Let x^1, \dots, x^m be local coordinates of M , and let z^1, \dots, z^{2n} be local coordinates of N . We denote

$$g = \sum_{i,j=1}^m g_{ij} dx^i \otimes dx^j, \quad \sum_{k=1}^m g_{ik} g^{kj} = \delta_{ij}, \quad G = \det(g_{ij}), \quad \Delta_g = \sum_{i,j=1}^m \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{G} \frac{\partial}{\partial x^j},$$

$$h = \sum_{a,b=1}^{2n} h_{ab} dz^a \otimes dz^b, \quad \sum_{c=1}^{2n} h_{ac} h^{cb} = \delta_{ab}, \quad \Gamma_{bc}^a = \frac{1}{2} \sum_{d=1}^{2n} h^{ad} \left(\frac{\partial h_{bd}}{\partial z^c} + \frac{\partial h_{cd}}{\partial z^b} - \frac{\partial h_{bc}}{\partial z^d} \right),$$

where δ_{ij} is Kronecker's delta. If we set $u^a = z^a \circ u$, the local expression of $\tau(u)$ is given by

$$\tau(u) = \sum_{i,j=1}^m g^{ij} \nabla du \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \sum_{a=1}^{2n} \left\{ \Delta_g u^a + \sum_{i,j=1}^m \sum_{b,c=1}^{2n} g^{ij}(x) \Gamma_{bc}^a(u) \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^j} \right\} \left(\frac{\partial}{\partial z^a} \right)_u.$$

Then, (1) is a $2n \times 2n$ system of quasilinear Schrödinger equations.

The equation (1) geometrically generalizes two-sphere valued partial differential equations modeling the motion of vertex filament, ferromagnetic spin chain system and etc. See, e.g., [4], [10], [27] and references therein. In the last decade, these physics models have been generalized and studied from a point of view of geometric analysis in mathematics. In other words, the relationship between

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the structure of the partial differential equation (1) and geometric settings have been investigated in the recent ten years. There are apparently two directions in the geometric analysis of partial differential equations like (1).

One of them is a geometric reduction of equations to simpler ones with values in the real or complex Euclidean space. This direction originated from Hasimoto's work [10]. In their pioneering work [1], Chang, Shatah and Uhlenbeck first rigorously studied the PDE structure of (1) when (M, g) is the real line with the usual metric and (N, J, h) is a compact Riemann surface. They constructed a good moving frame along the map and reduced (1) to a simple complex-valued equation when $u(t, x)$ has a fixed base point as $x \rightarrow +\infty$. Similarly, Onodera reduced a one-dimensional third or fourth order dispersive flow to a complex-valued equation in [25]. In [20] and [21], Nahmod, Stefanov and Uhlenbeck obtained a system of semilinear Schrödinger equations from the equation of the Schrödinger map of the Euclidean space to the two-sphere when the Schrödinger map never takes values in some open set of the two-sphere. Nahmod, Shatah, Vega and Zeng constructed a moving frame along the Schrödinger map of the Euclidean space to a Kähler manifold in [19]. Generally speaking, these reductions require some restrictions on the range of the mappings, and one cannot make use of them to solve the initial value problem for the original equations without restrictions on the range of the initial data.

The other direction of geometric analytic approach to partial differential equations like (1) is to consider how to solve the initial value problem. In his pioneering work [15], Koiso first reformulated the equation of the motion of vortex filament geometrically, and proposed the equation (1) when (M, g) is the one-dimensional torus and (N, J, h) is a compact Kähler manifold. Moreover, Koiso established the standard short-time existence theorem, and proved that if (N, J, h) is locally symmetric, that is, $\nabla^N R = 0$, then the solution exists globally in time, where ∇^N and R are the Levi-Civita connection and the Riemannian curvature tensor of N respectively. See [26] also. Similarly, Onodera studied local and global existence theorem of solutions to a third-order dispersive flow for closed curves into Kähler manifolds in [24]. Gustafon, Kang and Tsai studied time-global stability of the Schrödinger map of the two-dimensional Euclidean space to the two-sphere around equivariant harmonic maps in [9]. In [5], Ding and Wang gave a short-time existence theorem of (1)-(2) when (M, g) is a general closed Riemannian manifold and (N, J, h) is a compact Kähler manifold. However, they actually gave the proof only for the case that (M, g) is the Euclidean space or the flat torus. We do not know whether their method of proof can work for a general closed Riemannian manifold (M, g) . Generally speaking, Schrödinger evolution equations are very delicate on lower order terms, in contrast with the heat equations, which can be easily treated together with any lower order term by the classical Gårding inequality.

Both of these two directions of geometric analysis of equations like (1) are deeply concerned with the relationship between the geometric settings of equations and the theory of linear dispersive partial differential equations. For the latter subject, see, e.g., [3], [6], [12], [13], [17, Lecture VII] and references therein. Being concerned with the compactness of the source manifold, we need to mention local smoothing effect of dispersive partial differential equations. It is well-known that solutions to the initial value problem for some kinds of dispersive equations gain extra smoothness in comparison with the initial data. In his celebrated work [6], Doi characterized the existence of microlocal smoothing effect of Schrödinger evolution equations on complete Riemannian manifolds according to the global behavior of the geodesic flow on the unit cotangent sphere bundle over the source manifolds. Roughly speaking, the local smoothing effect occurs if and only if all the geodesics go to "infinity". In particular, if the source manifold is compact, then no smoothing effect occurs. For this reason, it is essential to study the initial value problem (1)-(2) when (M, g) is compact.

We should also mention the influence of the Kähler condition $\nabla^N J = 0$ on the structure of the equation (1). All the preceding works on (1) assume that (N, J, h) is a Kähler manifold. If $\nabla^N J \neq 0$,

then (1) behaves like symmetric hyperbolic systems, and the classical energy method works effectively. See [15] for the detail. If $\nabla^N J \neq 0$, then (1) has a first order terms in some sense, and the classical energy method breaks down.

The purpose of the present paper is to show a short-time existence theorem for (1)-(2) without the Kähler condition. To state our results, we here introduce function spaces of mappings. Set $\nabla_i = \nabla_{\partial/\partial x^i}$ for short. For a positive integer k , $H^k(M; TN)$ is the set of all continuous mappings $u : M \rightarrow TN$ satisfying

$$\|u\|_{H^k}^2 = \sum_{l=1}^k \int_M |\nabla^l u|^2 d\mu_g < \infty,$$

$$|\nabla^l u|^2 = \sum_{\substack{i_1, \dots, i_l \\ j_1, \dots, j_l=1}}^m g^{i_1 j_1} \dots g^{i_l j_l} h \left(\nabla^l u \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_l}} \right), \nabla^l u \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_l}} \right) \right),$$

where $d\mu_g = \sqrt{G} dx^1 \dots dx^m$ is the Riemannian measure of (M, g) . See e.g., [11] for the Sobolev space of mappings. The Nash embedding theorem shows that there exists an isometric embedding $w \in C^\infty(N; \mathbb{R}^d)$ with some integer $d > 2n$. See [7], [8] and [22]. Let I be an interval in \mathbb{R} . We denote by $C(I; H^k(M; TN))$ the set of all $H^k(M; TN)$ -valued continuous functions on I . In other words, we define it by the pullback of the function space by the isometry w as $C(I; H^k(M; TN)) = C(I; w^* H^k(M; \mathbb{R}^d))$, where $H^k(M; \mathbb{R}^d)$ is the usual Sobolev space of \mathbb{R}^d -valued functions on M .

Here we state our main results.

Theorem 1. *Let k be a positive integer satisfying $2k > m/2 + 5$, and let k_0 be the minimum of k . Then, for any $u_0 \in H^{2k}(M; TN)$, there exists $T = T(\|u_0\|_{H^{2k_0}}) > 0$ such that (1)-(2) possesses a unique solution $u \in C([-T, T]; H^{2k}(M; TN))$.*

Our strategy of proof consists of fourth order parabolic regularization and the uniform energy estimates of approximating solutions. Let $\Gamma(u^{-1}TN)$ be the set of sections of $u^{-1}TN$. Our idea of avoiding the difficulty due to $\nabla_i J_u$ comes from diagonalization technique for some $2n \times 2n$ system of Schrödinger evolution equations developed in our work [2]. If we see (1) as a $2n \times 2n$ system, $\nabla^N J$ corresponds to some off-diagonal blocks of the coefficient matrices of the first order terms. We introduce a bounded pseudodifferential operators acting on $\Gamma(u^{-1}TN)$ to eliminate $\nabla_i J_u$ by using a transformation of u with this pseudodifferential operator. We here remark that $(\nabla_{\partial/\partial z^a} J)J = -J(\nabla_{\partial/\partial z^a} J)$ since $J^2 = C_1^2 J \otimes J = -I$, where I is the identity mapping on TN and C_1^2 is a contraction of $(2, 2)$ -tensor. This fact is the key to construct the pseudodifferential operator. We evaluate $\tilde{\Delta}_g^{l-1} \tau(u)$, ($l = 1, \dots, k$), since

$$\tilde{\Delta}_g = \frac{1}{\sqrt{G}} \sum_{i,j=1}^m \nabla_i g^{ij} \sqrt{G} \nabla_j$$

commutes with $\tau(u)$ and is never an obstruction to the energy estimates. For this reason, we need to use even order Sobolev space $H^{2k}(M; TN)$. It is easy to check that $\tilde{\Delta}_g$ is invariant under the change of variables of M and N . Indeed, one can check that $\tilde{\Delta}_g$ is invariant under the change of variables of M in the same way as Δ_g , and $\nabla_i V$ is invariant under the change of variables of N for any section $V \in \Gamma(u^{-1}TN)$. Hence $\tilde{\Delta}_g$ is globally well-defined on $\Gamma(u^{-1}TN)$.

The plan of the present paper is as follows. Section 2 is devoted to parabolic regularization. In Section 3 we prove Theorem 1.

2. PARABOLIC REGULARIZATION

This section is devoted to the short-time existence of solutions to the forward initial value problem for a fourth order parabolic equations of the form

$$\frac{\partial u}{\partial t} = -\varepsilon \tilde{\Delta}_g \tau(u) + J_u \tau(u) \quad \text{in } (0, \infty) \times M, \quad (3)$$

$$u(0, x) = u_0(x) \quad \text{in } M, \quad (4)$$

where $\varepsilon \in (0, 1]$ is a parameter. Roughly speaking, (3)-(4) can be solved in the same way as the equation of harmonic heat flow $\partial u / \partial t = \tau(u)$. See e.g., [23, Chapters 3 and 4] for the study of the harmonic heat flow. In this section we shall show the following.

Lemma 2. *Let l be an integer satisfying $l \geq l_0 = [m/2] + 4$. Then, for any $u_0 \in H^l(M; TN)$, there exists $T_\varepsilon = T(\varepsilon, \|u_0\|_{H^{l_0}}) > 0$ such that the forward initial value problem (3)-(4) possesses a unique solution $u_\varepsilon \in C([0, T_\varepsilon]; H^l(M; TN))$.*

We consider \mathbb{R}^d -valued equation pushed by dw . We split the proof of Lemma 2 into two steps. Firstly, we construct a solution taking values in a tubular neighborhood of $w(N)$. Secondly, we check that the solution is $w(N)$ -valued.

If u is a solution to (3), then $v = w \circ u$ satisfies

$$\frac{\partial v}{\partial t} = dw \left(\frac{\partial u}{\partial t} \right) = dw \left(-\varepsilon \tilde{\Delta}_g \tau(u) + J_u \tau(u) \right) = -\varepsilon \Delta_g^2 v + F(v),$$

where $F(v)$ is of the form

$$F(v) = \varepsilon F_1(x, v, \bar{\nabla} v, \bar{\nabla}^2 v, \bar{\nabla}^3 v) + F_2(x, v, \bar{\nabla} v, \bar{\nabla}^2 v)$$

satisfying $F_1(x, y, 0, 0, 0) = 0$, $F_2(x, y, 0, 0) = 0$ for any $(x, y) \in M \times w(N)$, and $\bar{\nabla}$ is the connection induced by $v(t) : M \rightarrow w(N)$.

To solve this equation for v , we need elementary facts on the fundamental solution. Let $H^s(M) = (1 - \Delta_g)^{-s/2} L^2(M)$ be the usual Sobolev space on M of order $s \in \mathbb{R}$. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators of a Hilbert space \mathcal{H}_1 to a Hilbert space \mathcal{H}_2 . Set $\mathcal{L}(\mathcal{H}_1) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_1)$ for short. The existence and the properties of the fundamental solution are the following.

Lemma 3. *There exists an operator $E(t)$ satisfying*

$$E(t) \in C((0, \infty); \mathcal{L}(H^s(M))) \cap C^1((0, \infty); \mathcal{L}(H^{s+4}(M), H^s(M)))$$

for any $s \in \mathbb{R}$, such that

$$\left(\frac{\partial}{\partial t} + \varepsilon \Delta_g^2 \right) E(t) = 0 \quad \text{in } C((0, \infty); \mathcal{L}(H^{s+4}(M), H^s(M))),$$

$$\lim_{t \downarrow 0} E(t)v = v \quad \text{for any } v \in H^s(M)$$

$$\left\{ E(t)t^{3/4} \right\}_{t>0} \quad \text{is bounded in } \mathcal{L}(H^{s-3}(M), H^s(M)).$$

Lemma 3 is proved by the symbolic calculus of pseudodifferential operators. See [16, Chapter 7, Theorem 4.1] and [14].

Let $\delta > 0$ be a sufficiently small constant. We denote by $w_\delta(N)$ a tubular neighborhood of $w(N)$ in \mathbb{R}^d , that is,

$$w_\delta(N) = \left\{ v_1 + v_2 \in \mathbb{R}^d \mid v_1 \in w(N), v_2 \in T_{v_1} w(N)^\perp, |v_2| < \delta \right\},$$

where $|v_2| = \sqrt{(v_2^1)^2 + \dots + (v_2^d)^2}$ for $v_2 = (v_2^1, \dots, v_2^d) \in \mathbb{R}^d$. Let $\pi : w_\delta(N) \rightarrow w(N)$ be the projection defined by $\pi(v_1 + v_2) = v_1$ for $v_1 \in w(N)$, $v_2 \in T_{v_1} w(N)^\perp$, $|v_2| < \delta$. We will solve the initial value problem

$$\frac{\partial v}{\partial t} = -\varepsilon \Delta_g^2 v + F(\pi(v)), \quad v(0, x) = w(u_0)(x), \quad (5)$$

which is equivalent to an integral equation $v = \Phi(v)$, where

$$\Phi(v)(t) = E(t)w(u_0) + \int_0^t E(t-s)F(\pi(v(s)))ds.$$

We apply the contraction mapping theorem to the integral equation in the framework

$$X_T^l = \left\{ v \in C([0, T]; H^l(M; Tw_\delta(N))) \mid \|v(t)\|_{H^l} \leq 2\|w(u_0)\|_{H^l} \text{ for } t \in [0, T] \right\}$$

with some small $T > 0$ determined below. By using Lemma 3 and the Sobolev embeddings, we have

$$\begin{aligned} \sup_{t \in [0, T]} \|\Phi(v)(t) - E(t)w(u_0)\|_{H^l} &\leq C_l T^{1/4}, \\ \sup_{t \in [0, T]} \|\Phi(v)(t) - \Phi(v')(t)\|_{H^l} &\leq C_l T^{1/4} \sup_{t \in [0, T]} \|v(t) - v'(t)\|_{H^l} \end{aligned}$$

for any $v, v' \in X_T^l$, where C_l is a positive constant depending on $\|w(u_0)\|_{H^l}$. Thus, Φ is a contraction mapping of X_T^l to itself provided that T is sufficiently small. The contraction mapping theorem shows the existence of a unique solution to the integral equation.

Next we check that the solution $v \in C([0, T]; H^l(M; Tw_\delta(N)))$ to (5) is $w(N)$ -valued. Set $\rho(v) = v - \pi(v)$ for short. We remark that

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\varepsilon \Delta_g^2 v + F(\pi(v)) \\ &= -\varepsilon \Delta_g^2 \rho(v) - \varepsilon \Delta_g^2 \pi(v) + F(\pi(v)) \\ &= -\varepsilon \Delta_g^2 \rho(v) + dw \left(-\varepsilon \tilde{\Delta}_g \tau(w^{-1} \circ \pi(v)) + J_{w^{-1} \circ \pi(v)} \tau(w^{-1} \circ \pi(v)) \right). \end{aligned}$$

Since $\partial \pi(v) / \partial t \perp \rho(v)$ and $\rho(v) \in T_{\pi(v)} w(N)^\perp$, we deduce

$$\begin{aligned} \frac{d}{dt} \int_M |\rho|^2 d\mu_g &= 2 \int_M \left\langle \frac{\partial \rho(v)}{\partial t}, \rho(v) \right\rangle d\mu_g \\ &= 2 \int_M \left\langle \frac{\partial v}{\partial t}, \rho(v) \right\rangle d\mu_g \\ &= 2 \int_M \langle -\varepsilon \Delta_g^2 \rho(v) + dw(\dots), \rho(v) \rangle d\mu_g \\ &= -2\varepsilon \int_M \langle \Delta_g^2 \rho(v), \rho(v) \rangle d\mu_g \\ &= -2\varepsilon \int_M \langle \Delta_g \rho(v), \Delta_g \rho(v) \rangle d\mu_g \leq 0, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^d . We conclude that $\rho(v) = 0$ and therefore v is $w(N)$ -valued since $\rho(v(0)) = \rho(w(u_0)) = 0$. This completes the proof of Lemma 2.

3. UNIFORM ENERGY ESTIMATES

The present section proves Theorem 1 in three steps: existence, uniqueness and recovery of continuity. Let $u_\varepsilon \in C([0, T_\varepsilon]; H^{2k}(M; TN))$ be the unique solution to (3)-(4).

Existence. We shall show that there exists $T > 0$ which is independent of $\varepsilon \in (0, 1]$, such that $\{u_\varepsilon\}_{\varepsilon \in (0, 1]}$ is bounded in $L^\infty(0, T; H^{2k}(M; TN))$, which is the set of all H^{2k} -valued essentially bounded functions on $(0, T)$. If this is true, then the standard compactness argument shows that there exist u and a subsequence $\{u_\varepsilon\}$ such that

$$\begin{aligned} u_\varepsilon &\longrightarrow u \quad \text{in } C([0, T]; H^{2k-1}(M; TN)), \\ u_\varepsilon &\longrightarrow u \quad \text{in } L^\infty(0, T; H^{2k}(M; TN)) \quad \text{weakly star,} \end{aligned}$$

as $\varepsilon \downarrow 0$, and u solves (1)-(2) and is H^{2k} -valued weakly continuous in time.

For $V \in \Gamma(u^{-1}TN)$, set

$$\|V\|^2 = \int_M h(V, V) d\mu_g$$

for short. In view of the Sobolev embeddings, $\|u\|_{H^{2k}}$ is equivalent to

$$\left\{ \sum_{l=1}^k \|\tilde{\Delta}_g^l u\|^2 \right\}^{1/2}, \quad \tilde{\Delta}_g^l u = \tilde{\Delta}_g^{l-1} \tau(u),$$

as a norm. We shall evaluate this for u_ε instead of $\|u_\varepsilon\|_{H^{2k}}$.

The properties of the torsion tensor and the Riemannian curvature tensor show that for any vector fields X and Y on M , and for any $V \in \Gamma(u^{-1}TN)$,

$$\nabla_X du(Y) = \nabla_Y du(X) + du([X, Y]), \quad (6)$$

$$\nabla_X \nabla_Y V = \nabla_Y \nabla_X V + \nabla_{[X, Y]} V + R(du(X), du(Y))V. \quad (7)$$

Set $\nabla_t = \nabla_{\partial/\partial t}$, $X_i = \partial/\partial x^i$ and $Y_i = \sum_{j=1}^m g^{ij} X_j$ for short.

In what follows we express u_ε by u simply. Any confusion will not occur. We sometimes write $\tilde{\Delta}_g u = \tau(u)$ and $Xu = du(X)$. We evaluate $\mathcal{N}_k(u)$ defined by

$$\mathcal{N}_k(u)^2 = \sum_{j=1}^{k-1} \|\tilde{\Delta}_g^j u\|^2 + \|\Lambda \tilde{\Delta}_g^k u\|^2,$$

where $\Lambda = \Lambda_\varepsilon$ is a pseudodifferential operator defined later. Set

$$T_\varepsilon^* = \sup\{T > 0 \mid \mathcal{N}_k(u(t)) \leq 2\mathcal{N}_k(u_0) \text{ for } t \in [0, T]\}.$$

To obtain the uniformly estimates of $\mathcal{N}_k(u)$, we need to compute

$$\tilde{\Delta}_g^l \left(\frac{\partial u}{\partial t} + \varepsilon \tilde{\Delta}_g^2 u - J_u \tilde{\Delta}_g u \right) = 0, \quad l = 1, \dots, k.$$

In view of (6) and (7), we have

$$\begin{aligned} \tilde{\Delta}_g \frac{\partial u}{\partial t} &= \frac{1}{\sqrt{G}} \sum_{i=1} \nabla_{X_i} \left\{ \sqrt{G} \nabla_{Y_i} du(\partial/\partial t) \right\} \\ &= \frac{1}{\sqrt{G}} \sum_{i=1} \nabla_{X_i} \left\{ \nabla_t \sqrt{G} du(Y_i) + \sqrt{G} du([Y_i, \partial/\partial t]) \right\} \\ &= \frac{1}{\sqrt{G}} \sum_{i=1} \nabla_{X_i} \left\{ \nabla_t \sqrt{G} du(Y_i) \right\} \end{aligned}$$

$$\begin{aligned}
&= \nabla_t \left\{ \frac{1}{\sqrt{G}} \sum_{i=1}^m \nabla_{X_i} \sqrt{G} du(Y_i) \right\} \\
&\quad + \frac{1}{\sqrt{G}} \sum_{i=1}^m \nabla_{[X_i, \partial/\partial t]} \sqrt{G} du(Y_i) \\
&\quad + \sum_{i=1}^m R(du(X_i), du(\partial/\partial t)) du(Y_i) \\
&= \nabla_t \tilde{\Delta}_g u + \sum_{i=1}^m R(du(X_i), du(\partial/\partial t)) du(Y_i).
\end{aligned}$$

Repeating this computation and using (3), we obtain

$$\begin{aligned}
\tilde{\Delta}_g^l \frac{\partial u}{\partial t} &= \nabla_t \tilde{\Delta}_g^l u + \sum_{p=0}^{l-1} \tilde{\Delta}_g^{l-1-p} \left\{ R(du(X_i), du(\partial/\partial t)) \nabla_{Y_i} \tilde{\Delta}_g^p u \right\} \\
&= \nabla_t \tilde{\Delta}_g^l u + \sum_{p=0}^{l-1} \tilde{\Delta}_g^{l-1-p} \left\{ R(du(X_i), -\varepsilon \tilde{\Delta}_g^2 u + J_u \tilde{\Delta}_g u) \nabla_{Y_i} \tilde{\Delta}_g^p u \right\} \\
&= \nabla_t \tilde{\Delta}_g^l u - \varepsilon P_{1,l} - Q_{1,l}, \\
P_{1,l} &= \sum_{p=0}^{l-1} \tilde{\Delta}_g^{l-1-p} \left\{ R(du(X_i), \tilde{\Delta}_g^2 u) \nabla_{Y_i} \tilde{\Delta}_g^p u \right\}, \\
Q_{1,l} &= - \sum_{p=0}^{l-1} \tilde{\Delta}_g^{l-1-p} \left\{ R(du(X_i), J_u \tilde{\Delta}_g u) \nabla_{Y_i} \tilde{\Delta}_g^p u \right\}.
\end{aligned} \tag{8}$$

The Sobolev embeddings show that for $l = 1, \dots, k$, there exists a constant $C_k > 1$ depending only on $k \geq k_0$ and $\mathcal{N}_k(u_0)$ such that for $t \in [0, T_\varepsilon^*]$

$$\|P_{1,l}\| \leq C_k \sum_{p=1}^{l+1} \|\tilde{\Delta}_g^p u\|, \quad \|Q_{1,l}\| \leq C_k \sum_{p=1}^l \|\tilde{\Delta}_g^p u\|.$$

Different positive constants depending only on $k \geq k_0$ and $\mathcal{N}_k(u_0)$ are denoted by the same notation C_k below.

A direct computation shows that

$$\begin{aligned}
\tilde{\Delta}_g^l (J_u \tau(u)) &= \frac{1}{\sqrt{G}} \sum_{i=1}^m \nabla_{X_i} \left(J_u \sqrt{G} \nabla_{Y_i} \tilde{\Delta}_g^l u \right) \\
&\quad + l \sum_{i=1}^m (\nabla_{Y_i} J_u) \nabla_{X_i} \tilde{\Delta}_g^l u \\
&\quad + (l-1) \sum_{i=1}^m (\nabla_{X_i} J_u) \nabla_{Y_i} \tilde{\Delta}_g^l u + Q_{2,l} \\
&= \frac{1}{\sqrt{G}} \sum_{i,j=1}^m \nabla_i \left(g^{i,j} \sqrt{G} J_u \nabla_j \tilde{\Delta}_g^l u \right) \\
&\quad + (2l-1) \sum_{i,j=1}^m g^{ij} (\nabla_i J_u) \nabla_j \tilde{\Delta}_g^l u + Q_{2,l},
\end{aligned} \tag{9}$$

where $Q_{2,l}$ is a linear combination of terms of the form $(\nabla^{p+2}J_u)\nabla^{2l-p}u$, $p = 0, 1, \dots, 2l - 2$, and has the same estimate as $Q_{1,l}$.

Combining (8) and (9), we have

$$\left\{ \nabla_t + \varepsilon \tilde{\Delta}_g^2 - \frac{1}{\sqrt{G}} \sum_{i,j=1}^m \nabla_i g^{ij} \sqrt{G} J_u \nabla_j - (2l-1) \sum_{i,j=1}^m g^{ij} (\nabla_i J_u) \nabla_j \right\} \tilde{\Delta}_g^l u = \varepsilon P_{1,l} + Q_{1,l} + Q_{2,l}. \quad (10)$$

Set

$$P_{2,l} = (2l-1) \sum_{i,j=1}^m g^{ij} (\nabla_i J_u) \nabla_j \tilde{\Delta}_g^l u + \varepsilon P_{1,l} + Q_{1,l} + Q_{2,l}.$$

for short. $P_{2,l}$ can be estimated in the same way as $P_{1,l}$. Using (10), we deduce

$$\begin{aligned} \frac{d}{dt} \sum_{l=1}^{k-1} \|\tilde{\Delta}_g^l u\|^2 &= \frac{d}{dt} \sum_{l=1}^{k-1} \int_M h(\tilde{\Delta}_g^l u, \tilde{\Delta}_g^l u) d\mu_g \\ &= 2 \sum_{l=1}^{k-1} \int_M h(\nabla_t \tilde{\Delta}_g^l u, \tilde{\Delta}_g^l u) d\mu_g \\ &= -2\varepsilon \sum_{l=1}^{k-1} \int_M h(\tilde{\Delta}_g^2 \tilde{\Delta}_g^l u, \tilde{\Delta}_g^l u) d\mu_g \\ &\quad + 2 \sum_{l=1}^{k-1} \sum_{i,j=1}^m \int_M \frac{1}{\sqrt{G}} h(\nabla_i g^{ij} \sqrt{G} J_u \nabla_j \tilde{\Delta}_g^l u, \tilde{\Delta}_g^l u) d\mu_g \\ &\quad + 2 \sum_{l=1}^{k-1} \int_M h(P_{2,l}, \tilde{\Delta}_g^l u) d\mu_g \\ &= -2\varepsilon \sum_{l=1}^{k-1} \int_M h(\tilde{\Delta}_g^{l+1} u, \tilde{\Delta}_g^{l+1} u) d\mu_g \\ &\quad - 2 \sum_{l=1}^{k-1} \sum_{i,j=1}^m \int_M g^{ij} h(J_u \nabla_j \tilde{\Delta}_g^l u, \nabla_i \tilde{\Delta}_g^l u) d\mu_g \\ &\quad + 2 \sum_{l=1}^{k-1} \int_M h(P_{2,l}, \tilde{\Delta}_g^l u) d\mu_g \\ &= -2\varepsilon \sum_{l=1}^{k-1} \|\tilde{\Delta}_g^{l+1} u\|^2 + 2 \sum_{l=1}^{k-1} \int_M h(P_{2,l}, \tilde{\Delta}_g^l u) d\mu_g \\ &\leq 2C_k \sum_{l=1}^k \|\tilde{\Delta}_g^l u\|^2. \end{aligned} \quad (11)$$

To complete the energy estimates, we need to eliminate the first order term in (10). For this purpose, we here give the definition of the pseudodifferential operator Λ . Let $\{M_\nu\}$ be the finite set of local coordinate neighborhood of M , and let x_ν^1, \dots, x_ν^m be the local coordinates in M_ν . Let $\{N_\alpha\}$ be the set of local coordinate neighborhood of N , and let $z_\alpha^1, \dots, z_\alpha^{2n}$ be the local coordinates of N_α . We denote by C_0^∞ the set of all smooth functions with a compact support. Pick up partitions

of unity $\{\phi_\nu\} \subset C_0^\infty(M)$ and $\{\Phi_\alpha\} \subset C_0^\infty(N)$ subordinated to $\{M_\nu\}$ and $\{N_\alpha\}$ respectively. Take $\{\psi_\nu\}, \{\xi_\nu\} \subset C_0^\infty(N)$ and $\{\Psi_\alpha\}, \{\Xi_\alpha\} \subset C_0^\infty(N)$ so that

$$\begin{aligned} \xi_\nu &= 1 \quad \text{in} \quad \text{supp}[\phi_\nu], & \psi_\nu &= 1 \quad \text{in} \quad \text{supp}[\xi_\nu], & \text{supp}[\psi_\nu] &\subset M_\nu, \\ \Xi_\alpha &= 1 \quad \text{in} \quad \text{supp}[\Phi_\alpha], & \Psi_\alpha &= 1 \quad \text{in} \quad \text{supp}[\Xi_\alpha], & \text{supp}[\Psi_\alpha] &\subset N_\alpha. \end{aligned}$$

We define a local $(1, 1)$ -tensor by

$$B_{\nu, \alpha, j} = -(2k-1) \sum_{i=1}^m g^{ij} (\nabla_i J_u) \quad \text{in} \quad M_\nu \cap u^{-1}(N_\alpha).$$

Note that $J_u B_{\nu, \alpha, j} = -B_{\nu, \alpha, j} J_u$. Using the partitions of unity and the local tensors, we define a properly supported pseudodifferential operators of order -1 acting on $\Gamma(u^{-1}TN)$ by

$$\begin{aligned} \tilde{\Lambda} &= \frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_\nu(x) \Phi_\alpha(u) \sum_{j=1}^m J_u B_{\nu, \alpha, j} \nabla_j \xi_\nu(x) \Xi_\alpha(u) (1 - \Delta_g)^{-1} \psi_\nu(x) \Psi_\alpha(u) \\ &\quad - \frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_\nu(x) \Phi_\alpha(u) \sum_{j=1}^m B_{\nu, \alpha, j} J_u \nabla_j \xi_\nu(x) \Xi_\alpha(u) (1 - \Delta_g)^{-1} \psi_\nu(x) \Psi_\alpha(u), \end{aligned}$$

where ∇_j is expressed by the coordinates x_ν^1, \dots, x_ν^m and $z_\alpha^1, \dots, z_\alpha^{2n}$. Here we remark that for

$$V = \sum_{a=1}^{2n} V_{\nu, \alpha}^a \left(\frac{\partial}{\partial z_\alpha^a} \right)_u$$

supported in $M_\nu \cap u^{-1}(N_\alpha)$,

$$\xi_\nu(x) \Xi_\alpha(u) (1 - \Delta_g)^{-1} V = \sum_{a=1}^{2n} (\xi_\nu(x) \Xi_\alpha(u) (1 - \Delta_g)^{-1} V_{\nu, \alpha}^a) \left(\frac{\partial}{\partial z_\alpha^a} \right)_u$$

is well-defined and supported in $M_\nu \cap u^{-1}(N_\alpha)$ also. We make use of elementary theory of pseudodifferential operators freely. We remark that each term in $\tilde{\Lambda}$ is properly supported and invariant under the change of coordinates in M and N up to cut-off functions. Then, we can deal with each term as if it were a pseudodifferential operator on \mathbb{R}^m acting on \mathbb{R}^d -valued functions. Then, we can make use of pseudodifferential operators whose symbols have limited smoothness. See [2, Section 2] and [18]. In other words, we do not have to take care of the type of this pseudodifferential operator. It is well-known that the type of pseudodifferential operators on manifolds have some restrictions in general.

Set $\Lambda = I - \tilde{\Lambda}$ and $\Lambda' = I + \tilde{\Lambda}$, where I is the identity mapping. Since $\Lambda' \Lambda = I - \tilde{\Lambda}^2$ and $\tilde{\Lambda}^2$ is a pseudodifferential operator of order -2 , we deduce that for $t \in [0, T_\varepsilon^*]$,

$$C_k^{-1} \mathcal{N}_k(u)^2 \leq \sum_{l=1}^k \|\tilde{\Delta}_g^l u\|^2 \leq C_k \mathcal{N}_k(u)^2.$$

We apply Λ to (10) with $l = k$. A direct computation shows that

$$\Lambda \nabla_t = \nabla_t \Lambda - \frac{\partial \Lambda}{\partial t}, \quad \left\| \frac{\partial \Lambda}{\partial t} \tilde{\Delta}_g^k u \right\| \leq C_k \mathcal{N}_k(u)$$

for $t \in [0, T_\varepsilon^*]$. The matrices of principal symbols of Λ and $\tilde{\Delta}_g^2$ commute with each other since the matrix of the principal symbol of $\tilde{\Delta}_g^2$ is $(g^{ij} \xi_i \xi_j)^2 I_{2n}$, where I_{2n} is the $2n \times 2n$ identity matrix. Then, $[\Lambda, \tilde{\Delta}_g^2] = [\tilde{\Lambda}, \tilde{\Delta}_g^2]$ is a pseudodifferential operator of order 2. Hence we have

$$\Lambda \tilde{\Delta}_g^2 = \tilde{\Delta}_g^2 \Lambda + [\Lambda, \tilde{\Delta}_g^2], \quad \|[\Lambda, \tilde{\Delta}_g^2] \tilde{\Delta}_g^k u\| \leq C_k \|\tilde{\Delta}_g^{k+1} u\|$$

for $t \in [0, T_\varepsilon^*]$.

Here we set

$$A = \sum_{i,j=1}^m \frac{1}{\sqrt{G}} \nabla_i g^{ij} \sqrt{G} J_u \nabla_j$$

for short. Since $I = \Lambda' \Lambda + \tilde{\Lambda}^2$, we deduce that

$$\Lambda(-A) = -\Lambda A(\Lambda' \Lambda + \tilde{\Lambda}^2) = -A + \tilde{\Lambda} A - A \tilde{\Lambda} + (\tilde{\Lambda} A \tilde{\Lambda} + A \tilde{\Lambda}^2),$$

and $\tilde{\Lambda} A \tilde{\Lambda} + A \tilde{\Lambda}^2$ is L^2 -bounded for $t \in [0, T_\varepsilon^*]$. The principal symbol of $(1 - \Delta_g)^{-1}$ is globally defined as $I_{2n}/g^{ij} \xi_i \xi_j$. We deduce that modulo L^2 -bounded operators,

$$\begin{aligned} \tilde{\Lambda} A &\equiv -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m B_{\nu, \alpha, j} J_u^2 \nabla_j \xi_{\nu}(x) \Xi_{\alpha}(u) (1 - \Delta_g)^{-1} \psi_{\nu}(x) \Psi_{\alpha}(u) \tilde{\Delta}_g \\ &\equiv -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m B_{\nu, \alpha, j} \nabla_j \xi_{\nu}(x) \Xi_{\alpha}(u) \psi_{\nu}(x) \Psi_{\alpha}(u) \\ &= -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m B_{\nu, \alpha, j} \nabla_j \\ &= \frac{(2k-1)}{2} \sum_{i,j} g^{ij} (\nabla_i J_u) \nabla_j, \\ -A \tilde{\Lambda} &\equiv -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \tilde{\Delta}_g \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m J_u^2 B_{\nu, \alpha, j} \nabla_j \xi_{\nu}(x) \Xi_{\alpha}(u) (1 - \Delta_g)^{-1} \psi_{\nu}(x) \Psi_{\alpha}(u) \\ &\equiv -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m B_{\nu, \alpha, j} \nabla_j \xi_{\nu}(x) \Xi_{\alpha}(u) \psi_{\nu}(x) \Psi_{\alpha}(u) \\ &= -\frac{1}{2} \sum_{\nu} \sum_{\alpha} \phi_{\nu}(x) \Phi_{\alpha}(u) \sum_{j=1}^m B_{\nu, \alpha, j} \nabla_j \\ &= \frac{(2k-1)}{2} \sum_{i,j} g^{ij} (\nabla_i J_u) \nabla_j. \end{aligned}$$

Combining computations above, we obtain

$$\left\{ \nabla_t + \varepsilon \tilde{\Delta}_g^2 - \frac{1}{\sqrt{G}} \sum_{i,j=1}^m \nabla_i g^{i,j} \sqrt{G} J_u \nabla_j \right\} \Lambda \tilde{\Delta}_g^k u = \varepsilon P_k + Q_k,$$

where P_k and Q_k are estimated as

$$\|P_k\| \leq C_k (\|\tilde{\Delta}_g \Lambda \tilde{\Delta}_g^k u\| + \mathcal{N}_k(u)), \quad \|Q_k\| \leq C_k \mathcal{N}_k(u).$$

In the same way as (11), we deduce

$$\frac{d}{dt} \|\Lambda \tilde{\Delta}_g^k u\|^2 \leq -2\varepsilon \|\tilde{\Delta}_g \Lambda \tilde{\Delta}_g^k u\|^2 + C_k \varepsilon \|\tilde{\Delta}_g \Lambda \tilde{\Delta}_g^k u\| \|\Lambda \tilde{\Delta}_g^k u\| + C_k \mathcal{N}_k(u) \|\Lambda \tilde{\Delta}_g^k u\| \quad (12)$$

$$\leq C_k \mathcal{N}_k(u) \|\Lambda \tilde{\Delta}_g^k u\|. \quad (13)$$

Combining (11) and (13), we obtain

$$\frac{d}{dt} \mathcal{N}_k(u) \leq C_k \mathcal{N}_k(u) \quad \text{for } t \in [0, T_\varepsilon^*]. \quad (14)$$

If we take $t = T_\varepsilon^*$, then we have $2\mathcal{N}_k(u_0) \leq \mathcal{N}_k(u_0)e^{C_k T_\varepsilon^*}$, which implies that $T_\varepsilon^* \geq T = \log 2/C_k > 0$. Thus $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^{2k}(M; TN))$. This completes the proof. \square

Uniqueness. Let $u_1, u_2 \in L^\infty(0, T; H^{2k}(M; TN))$ be solutions to (1)-(2). Set $v_1 = w \circ u_1$ and $v_2 = w \circ u_2$ for short. We denote by $\Pi_v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ the projection to $T_v w(N)$ for $v \in w(N)$. Since

$$\frac{\partial v}{\partial t} = \tilde{J}_{v_j} \Pi_{v_j} \Delta_g v_j, \quad \tilde{J}_{v_j} = du_j \circ J_{u_j} \circ du_j^{-1}, \quad j = 1, 2,$$

$v = v_1 - v_2$ solves

$$\begin{aligned} \frac{\partial v}{\partial t} &= \tilde{J}_{v_1} \Pi_{v_1} \Delta_g v + \left(\tilde{J}_{v_1} \Pi_{v_1} - \tilde{J}_{v_2} \Pi_{v_2} \right) \Delta_g v_2 \\ &= \tilde{J}_{v_1} \Pi_{v_1} \Delta_g v + A(v_1, v_2, \Delta_g v_2) v. \end{aligned}$$

Here we applied the mean value theorem to the second term of the right hand side of the above equation, and $A(v_1, v_2, \Delta_g v_2)$ is an appropriate $d \times d$ matrix.

Using the integration by parts, we have

$$\begin{aligned} \frac{d}{dt} \int_M \langle v, v \rangle d\mu_g &= 2 \int_M \left\langle \frac{\partial v}{\partial t}, v \right\rangle d\mu_g \\ &= 2 \int_M \left\langle \tilde{J}_{v_1} \Pi_{v_1} \Delta_g v + A(v_1, v_2, \Delta_g v_2) v, v \right\rangle d\mu_g \\ &\leq C \int_M \left\{ \langle v, v \rangle + \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle \right\} d\mu_g. \end{aligned} \quad (15)$$

Using the properties of \tilde{J}_{v_1} and the projection, and the integration by parts, we deduce

$$\begin{aligned} \frac{d}{dt} \int_M \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle d\mu_g &= 2 \int_M \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial^2 v}{\partial t \partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle d\mu_g \\ &= -2 \int_M \left\langle \frac{\partial v}{\partial t}, \Delta_g v \right\rangle d\mu_g \\ &= -2 \int_M \left\langle \tilde{J}_{v_1} \Pi_{v_1} \Delta_g v + A v, \Delta_g v \right\rangle d\mu_g \\ &= -2 \int_M \left\langle \tilde{J}_{v_1} \Pi_{v_1} \Delta_g v, \Pi_{v_1} \Delta_g v \right\rangle d\mu_g \\ &\quad - 2 \int_M \langle A v, \Delta_g v \rangle d\mu_g \\ &= -2 \int_M \langle A v, \Delta_g v \rangle d\mu_g \\ &\leq C \int_M \left\{ \langle v, v \rangle + \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle \right\} d\mu_g. \end{aligned} \quad (16)$$

Combining (15) and (16), we have

$$\frac{d}{dt} \int_M \left\{ \langle v, v \rangle + \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle \right\} d\mu_g \leq C \int_M \left\{ \langle v, v \rangle + \sum_{i,j=1}^m g^{ij} \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^j} \right\rangle \right\} d\mu_g,$$

which implies $v = 0$. This completes the proof. \square

Continuity in time. Let $u \in L^\infty(0, T; H^{2k}(M; TN))$ be the unique solution to (1)-(2). We remark that $u \in C([0, T]; H^{2k-1}(M; TN))$ and $\tilde{\Delta}_g^k u$ is a weakly continuous $L^2(M; TN)$ -valued function on $[0, T]$. We identify N and $w(N)$ below. Let $\{u_\varepsilon\}_{\varepsilon \in (0, 1]}$ be a sequence of solutions to (3)-(4), which approximates u . We can easily check that for any $\phi \in C^\infty([0, T] \times M; \mathbb{R}^d)$,

$$\begin{aligned} \Lambda_\varepsilon \phi &\longrightarrow \Lambda \phi && \text{in } L^2((0, T) \times M; \mathbb{R}^d), \\ u_\varepsilon &\longrightarrow u && \text{in } L^2((0, T) \times M; \mathbb{R}^d), \\ \Lambda_\varepsilon \tilde{\Delta}_g^k u_\varepsilon &\longrightarrow \tilde{u} && \text{in } L^2((0, T) \times M; \mathbb{R}^d) \text{ weakly star,} \end{aligned}$$

as $\varepsilon \downarrow 0$ with some \tilde{u} . Then, $\tilde{u} = \Lambda \tilde{\Delta}_g^k u$ in the sense of distributions. The time-continuity of $\tilde{\Delta}_g^k u$ is equivalent to that of $\Lambda \tilde{\Delta}_g^k u$ since $\Lambda \in C([0, T]; \mathcal{L}(L^2(M; \mathbb{R}^d)))$,

It suffices to show that

$$\lim_{t \downarrow 0} \Lambda(t) \tilde{\Delta}_g^k u(t) = \Lambda(0) \tilde{\Delta}_g^k u_0 \quad \text{in } L^2(M; \mathbb{R}^d), \quad (17)$$

since the other cases can be proved in the same way. (14) and the lower semicontinuity of L^2 -norm imply

$$\sum_{l=1}^{k-1} \|\tilde{\Delta}_g^l u(t)\|^2 + \|\Lambda(t) \tilde{\Delta}_g^k u(t)\|^2 \leq \sum_{l=1}^{k-1} \|\tilde{\Delta}_g^l u_0\|^2 + \|\Lambda(0) \tilde{\Delta}_g^k u_0\|^2 + C_k \mathcal{N}_k(u_0)^2 t.$$

Letting $t \downarrow 0$, we have

$$\limsup_{t \downarrow 0} \|\Lambda(t) \tilde{\Delta}_g^k u(t)\|^2 \leq \|\Lambda(0) \tilde{\Delta}_g^k u_0\|^2,$$

which implies (17). This completes the proof. \square

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